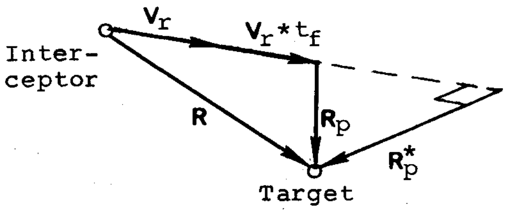

Fig. 2 Construction for predicted miss distance.

prediction is that more accurate measures of actual miss distance are obtained by projection than by interpolation from interceptor and target positions at discrete times. The required geometry is shown in Fig. 2. The predicted range vector $\boldsymbol{R}_p$ is found by projecting current positions and relative velocity $\boldsymbol{V}_r$ forward in time by t_f.

$$\boldsymbol{R}_p = \boldsymbol{R} - \boldsymbol{V}_r t_f \tag{8}$$

The minimum value of R_p or R_p^* is the closest approach. This occurs when the derivative of R_p^2 with respect to t_f vanishes. Squaring Eq. (8) gives

$$R_p^2 = R^2 - 2\boldsymbol{R}\cdot \boldsymbol{V}_r t_f + V_r^2 t_f^2$$

and

$$\partial R_p^2/\partial t_f = -2\boldsymbol{R}\cdot \boldsymbol{V}_r + 2V_r^2 t_f \tag{9}$$

setting Eq. (9) to zero

$$t_f^* = \boldsymbol{R}\cdot \boldsymbol{V}_r / V_r^2 \tag{10}$$

Using this result in Eq. (8), the predicted miss distance vector is

$$\boldsymbol{R}_p^* = \boldsymbol{R} - \boldsymbol{V}_r(\boldsymbol{R}\cdot \boldsymbol{V}_r)/V_r^2 \tag{11}$$

Equation (10) is easily reduced to components as

$$t_f^* = (x\dot{x} + y\dot{y} + z\dot{z})/(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) \tag{12}$$

The predicted miss distance R_p^* is the length of the vector given in Eq. (11), found by taking the square root of the squares of the components of the right-hand side or

$$R_p^* = \left[\frac{(y\dot{x} - x\dot{y})^2 + (z\dot{y} - y\dot{z})^2 + (x\dot{z} - z\dot{x})^2}{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}\right]^{1/2} \tag{13}$$

Numerical Example

Table 1 shows the end-game kinematics of a vehicle maneuvering under the PN law of Eq. (2) to intercept a reentry vehicle. At the last time point of 20.32 s, the interceptor has passed by the target and range is increasing. Average closing velocity is 16,500 fps. The minimum range point clearly cannot be found from the rapidly changing tabular values. On the other hand, the predicted miss distances from Eq. (13) are nearly constant at 11.0 ft. This is in the face of large and variable load factors.

Concluding Remarks

Proportional navigation for homing guidance is shown to have a vector formulation in addition to the classical scalar PN law. The vector formulation is suitable to the command guidance problem when line-of-sight rates can be deduced from interceptor and target positions and velocities in a set of inertial coordinates. Predicted miss distance derived from these same quantities converges satisfactorily in an interception at a high closing velocity.

Quaternion Singularity Revisited

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Introduction

In a 1970 paper, Grubin[1] proposed an algorithm for extracting a quaternion from a given direction cosine matrix (DCM). The algorithm is relatively simple but is singular when the rotation angle is 180 deg. Recently, Klumpp[2] and Shepperd[3] have proposed algorithms that are singularity-free but more complex. The purpose of this Note is 1) to examine numerically the range of validity of Grubin's algorithm and demonstrate that it can be valid for rotation angles from 0 to 180 deg $-\epsilon$, where ϵ is as small as a few hundredths of a degree, and 2) propose an alternate algorithm for handling the singular case.

Analysis

Given a DCM that relates two rectangular frames, the problem is to extract the corresponding quaternion. Let the DCM elements be a_{ij} $(i,j=1,2,3)$. Equations (22) of Ref. 1 determine the quaternion elements ξ,η,ζ,χ as

$$\begin{aligned}
\chi &= \cos(\theta/2) = (1+T)^{1/2}/2\\
\xi &= l\sin(\theta/2) = (a_{23}-a_{32})/4\chi\\
\eta &= m\sin(\theta/2) = (a_{31}-a_{13})/4\chi\\
\zeta &= n\sin(\theta/2)\,(a_{12}-a_{21})/4\chi
\end{aligned} \tag{1}$$

where $T = a_{11}+a_{22}+a_{33}$. In Eqs. (1) l, m, n, θ are the Euler axis/angle as described in Ref. 1.

So long as χ is not too small, Eqs. (1) are valid expressions for ξ, η, ζ. The singularity occurs for $\chi\to 0$ $(T\to -1)$, corresponding to $\theta\to 180$ deg. However, whenever this occurs, simultaneously $a_{23}\to a_{32}$, $a_{31}\to a_{13}$, $a_{12}\to a_{21}$, so that ξ, η, $\zeta\to 0/0=$? The question is, for what values of θ do Eqs. (1) begin to deteriorate?

Equations (7) and (19) of Ref. 1 give the DCM in terms of the Euler axis/angle parameters and the quaternion parameters, respectively. For the numerical studies, various sets of the Euler parameters were chosen, thereby generating values of the a_{ij}. The a_{ij} were then used in Eqs. (1) to determine the quaternion parameters. These, in turn, were substituted into Eqs. (19) of Ref. 1 to determine the reconstructed DCM elements $\hat{a}_{ij}$. The magnitude of the error

$$\epsilon_{ij} = |\hat{a}_{ij} - a_{ij}| \qquad (i,j=1,2,3)$$

was the quantity of interest.

Values of a switching angle, θ_s, were sought such that for $0\le\theta<\theta_s$ the errors would be "small." Also, for θ in the range $\theta_s\le\theta\le 180$ deg, alternate nonsingular expressions for ξ, η, ζ had to be developed. Ideally, θ_s should be such that maximum values for ϵ_{ij} would be equal for θ values slightly less than θ_s and slightly greater than θ_s.

Using a computer with 15-significant-figure accuracy and switching when $\chi = 2\times 10^{-4}$ gives $\theta_s = 179.9770817$ deg. Choosing θ slightly less, e.g., 179.9770807 deg and following the procedure described gave $\epsilon_{ij,\max} < 1\times 10^{-7}$. Again, choosing θ slightly greater than θ_s, e.g., 179.9770827 deg, and using the alternate expressions for ξ, η, ζ (given below), $\epsilon_{ij,\max}$ was again $<1\times 10^{-7}$. These results were verified for many

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directions of the Euler axis. Thus, for 15-significant-figure accuracy, θ_s given above is essentially an optimal value in the sense of roughly equalizing the maximum errors. If θ is a random variable uniformly distributed in the range 0 to 180, then $(179.9770817/180) = 0.99987$, so that for this fraction of the calculations Eqs. (1) can be used for computing the quaternion elements, with the $\epsilon_{ij,\max}$ being less than 1×10^{-7}. Only for the remaining 0.00013 of the calculations would the alternate expressions be required.

To obtain alternate expressions for ξ, η, ζ, χ that are valid for θ in the range $\theta_s \le \theta \le 180$ deg, use Eqs. (7), (18) and (22) of Ref. 1. For θ in the neighborhood of 180 deg, Eq. (7) of Ref. 1 gives the DCM very nearly as

$$\begin{bmatrix} 2l^2-1 & 2lm & 2ln \\ 2lm & 2m^2-1 & 2mn \\ 2ln & 2mn & 2n^2-1 \end{bmatrix} \tag{2}$$

while Eqs. (18) show that very nearly

$$\xi = l \qquad \eta = m \qquad \zeta = n \tag{3}$$

Combining Eqs. (2) and (3) produces

$$\begin{aligned} &2\xi^2 - 1 = a_{11} \qquad 2\eta^2 - 1 = a_{22} \qquad 2\zeta^2 - 1 = a_{33} \\ &2\xi\eta = a_{12} = a_{21} \qquad 2\xi\zeta = a_{13} = a_{31} \qquad 2\eta\zeta = a_{23} = a_{32} \end{aligned} \tag{4}$$

Equations (4) are nine equations for the three unknowns ξ, η, ζ, so there is considerable redundancy for their solution. One approach is to choose the largest value among a_{11}, a_{22}, and a_{33} and then solve for the corresponding ξ or η or ζ. The sign of the variable can be chosen positive, and then the magnitude and sign of the remaining terms are determined by solving the "cross-term" expressions. There is no real requirement for using the largest value; e.g., it would be satisfactory to solve for ξ if a_{11} were greater than some minimum value or solve for η if a_{22} were greater than some minimum value. However, using the largest value results in uniform logic and a fixed computing time. For χ, its magnitude is always given by Eq. (1) of the present paper, while its sign is chosen by observing Eqs (22b, c, d) of Ref. 1. These are

$$\xi\chi = (a_{23} - a_{32})/4 \qquad \eta\chi = (a_{31} - a_{13})/4 \qquad \zeta\chi = (a_{12} - a_{21})/4 \tag{5}$$

Thus, for example, if a_{11} is largest, so that ξ is determined from Eqs. (4) as

$$\xi = [(1 + a_{11})/2]^{1/2}$$

then

$$\operatorname{sgn}\chi = \operatorname{sgn}(a_{23} - a_{32})$$

where sgn $U = 1$, $U > 0$; sgn$U = 0$, $U = 0$; sgn$U = -1$, $U < 0$.

Alternatively, the sign of χ can be taken always positive and then

$$\operatorname{sgn}\xi = \operatorname{sgn}(a_{23} - a_{32})$$

The former rule is used in the equations below. The interchangeability of signs is a consequence of the fact that the DCM expressed in terms of the quaternion parameters consists of products of all terms [Eq. (19) of Ref. 1].

Combining all of the above then, the alternate expressions are:

If $\chi < 2 \times 10^{-4}$, choose the largest among a_{11}, a_{22}, a_{33}.

If a_{11} is largest,

$$\chi \leftarrow \chi \operatorname{sgn}(a_{23} - a_{32})$$

$$\xi = [(1 + a_{11})/2]^{1/2} \qquad \eta = (a_{12} + a_{21})/4\xi \qquad \zeta = (a_{13} + a_{31})/4\xi \tag{6a}$$

If a_{22} is largest,

$$\chi \leftarrow \chi \operatorname{sgn}(a_{31} - a_{13})$$

$$\eta = [(1 + a_{22})/2]^{1/2} \qquad \xi = (a_{12} + a_{21})/4\eta \qquad \zeta = (a_{23} + a_{32})/4\eta \tag{6b}$$

If a_{33} is largest,

$$\chi \leftarrow \chi \operatorname{sgn}(a_{12} - a_{21})$$

$$\zeta = [(1 + a_{33})/2]^{1/2} \qquad \xi = (a_{13} + a_{31})/4\zeta \qquad \eta = (a_{23} + a_{32})/4\zeta \tag{6c}$$

The errors incurred in using Eqs. (6a-c) arise, of course, from the fact that Eqs. (2) and (3) are strictly valid only for $\theta = 180$ deg. For $\theta = 180$ deg $- \epsilon$, some small error is incurred.

Conclusions

Assuming the rotation angle θ is a uniformly distributed random variable in the range $0 \le \theta \le 180$ deg and using 15-significant-figure accuracy, Eqs. (1) would be used for 99.987% of the quaternion extractions with maximum error (in the sense defined) $< 1 \times 10^{-7}$. For the remainder of the calculations, the alternate Eqs. (6) would be used, with approximately the same maximum error. Thus, for some applications, Eqs. (1) by themselves, without Eqs. (6), can be valid for all cases of interest. Conversely, if θ lies within an ϵ of 180 deg so that Eqs. (6) would also be needed, Refs. 2 and 3 should be considered since their algorithms use a single set of equations for the entire range $0 \le \theta \le 180$ deg. Naturally, if a computer with other than 15-significant-figure accuracy is available, the results given herein are subject to change.

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